

Generalized boson algebra and its entangled bipartite coherent states

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Abstract

Starting with a given generalized boson algebra $\mathcal{U}_{\langle q \rangle}(h(1))$ known as the bosonized version of the quantum super-Hopf $\mathcal{U}_q(osp(1|2))$ algebra, we employ the Hopf duality arguments to provide the dually conjugate function algebra $\mathcal{F}un_{\langle q \rangle}(H(1))$. Both the Hopf algebras being finitely generated, we produce a closed form expression of the universal \mathcal{T} matrix that caps the duality and generalizes the familiar exponential map relating a Lie algebra with its corresponding group. Subsequently, using an inverse Mellin transform approach, the coherent states of single-node systems subject to the $\mathcal{U}_{\langle q \rangle}(h(1))$ symmetry are found to be complete with a positive-definite integration measure. Nonclassical coalgebraic structure of the $\mathcal{U}_{\langle q \rangle}(h(1))$ algebra is found to generate naturally *entangled* coherent states in bipartite composite systems.

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I. Introduction

Quantization of the boson algebra has been actively investigated^{1–15} due to its importance in studies of quantum groups, special functions, integrable models and the theory of noncommuting spaces. Many recent works in this area^{9–14} focus on the quantized boson algebras endowed with Hopf structures as they are naturally equipped for applications in many-body systems of interest. In particular, it was observed by Macfarlane and Majid¹⁰ that a quantum boson algebra admitting a Hopf structure plays the role of the spectrum generating algebra for the q -oscillator,^{1–3} as it is the bosonized version of the super-Hopf $\mathcal{U}_q(osp(1|2))$ algebra. This algebra was further generalized and studied in Refs. [12, 13]. These authors also pointed out the close relation of this algebra with the Calogero-Sutherland^{16, 17} type of models. In the present work, we study and make applications of this generalized boson algebra $\mathcal{U}_{(q)}(h(1))$ defined in (2.1) and (2.2).

Using a technique developed by Fronsdal and Galindo¹⁸ in the context of $\mathcal{U}_q(gl(2))$ algebra, we in Sec. II. study the Hopf duality and obtain the full Hopf structure of the function algebra $\mathcal{F}un_{(q)}(H(1))$, dually conjugate to the $\mathcal{U}_{(q)}(h(1))$ algebra. The corresponding dual form, alternately referred to as the universal \mathcal{T} matrix, caps the duality structure and embodies the suitably modified exponential relationship $\mathcal{U}_{(q)}(h(1)) \rightarrow \mathcal{F}un_{(q)}(H(1))$. Noticing that both the Hopf algebras are finitely generated, we derive a closed form expression of the universal \mathcal{T} matrix in terms of two sets of generators. The main usefulness of the universal \mathcal{T} matrix stems from the fact that the transfer matrices of integrable models appear, upon specialization, in passing from operator structure to representations.

Enroute to our construction of the coherent states of the bipartite composite systems governed by $\mathcal{U}_{(q)}(h(1))$ symmetry, we in Sec. III. provide a resolution of unity via the coherent states of the corresponding single-node systems. This property allows the coherent states to be complete (actually, overcomplete) set, and this is essential for a majority of applications in quantum mechanics. Recent works^{19–22} in establishing the resolution of the unit operator in an ensemble of generalized coherent states have used the method of inverse Mellin transform. Using an inverse Mellin transform of an associated Stieltjes moment problem, we obtain the resolution of unity of the single-node coherent states in the form of an ordinary integral with a positive-definite measure.

Turning towards applications of the Hopf coalgebraic structure of the $\mathcal{U}_{(q)}(h(1))$ algebra we note that it leads to qualitatively new properties of the many-body systems. In Sec. IV. we introduce and analytically obtain the coherent states in a bipartite composite system subject to $\mathcal{U}_{(q)}(h(1))$ symmetry. The normalizable coherent states are *naturally entangled for a nonclassical value of q ($\neq 1$)*. The entanglement disappears in the classical $q \rightarrow 1$ limit. Study of quantum information theory using entangled coherent states is of much current interest.^{23, 24} Recently bipartite Barut-Girardello²⁵ coherent states of the $\mathcal{U}_q(su(1, 1))$ algebra have been found to be entangled for $q \neq 1$. Our present calculation adds to the expectation that the *entanglement of bipartite and multipartite coherent states is a generic feature of the quantum algebras*.

II. The dual algebra and the universal \mathcal{T} matrix

Following the authors in Refs. [10, 12, 13], we consider a q -deformed generalized boson algebra $\mathcal{U}_{(q)}(h(1))$ generated by a , a^\dagger and N subject to the commutation relations

$$aa^\dagger + a^\dagger a = [\alpha N + \beta]_q, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad (2.1)$$

where q has generic real value, $(\alpha, \beta) \in \mathbb{R}$, and $[\mathcal{X}]_q = \frac{q^\mathcal{X} - q^{-\mathcal{X}}}{q - q^{-1}}$. The supplementary generating element $g \equiv (-1)^{\tilde{N}}$, where $\tilde{N} = N + \frac{\beta}{\alpha} \mathbb{I}$, $\alpha \neq 0$, plays a key role in the construction of the Hopf coalgebraic structure. The coalgebraic maps read

$$\begin{aligned} \Delta(N) &= N \otimes \mathbb{I} + \mathbb{I} \otimes N + \frac{\beta}{\alpha} \mathbb{I} \otimes \mathbb{I}, \quad \Delta((-1)^{\tilde{N}}) = (-1)^{\tilde{N}} \otimes (-1)^{\tilde{N}}, \\ \Delta(a) &= a \otimes q^{\frac{\alpha \tilde{N}}{2}} + (-1)^{\tilde{N}} q^{-\frac{\alpha \tilde{N}}{2}} \otimes a, \quad \Delta(a^\dagger) = a^\dagger \otimes q^{\frac{\alpha \tilde{N}}{2}} + (-1)^{-\tilde{N}} q^{-\frac{\alpha \tilde{N}}{2}} \otimes a^\dagger, \\ \epsilon(N) &= -\frac{\beta}{\alpha}, \quad \epsilon((-1)^{\tilde{N}}) = 1, \quad \epsilon(a) = \epsilon(a^\dagger) = 0, \\ S(N) &= -N - \frac{2\beta}{\alpha}, \quad S(a) = -(-1)^{-\tilde{N}} q^{-\alpha/2} a, \quad S(a^\dagger) = a^\dagger (-1)^{\tilde{N}} q^{\alpha/2}. \end{aligned} \quad (2.2)$$

Imposing the constraint $g^2 = 1$, it has been found ¹⁰ that the algebra $\mathcal{U}_{(q)}(h(1))$ is the bosonized version of the super-Hopf $\mathcal{U}_q(osp(1|2))$ algebra. As it is endowed with bosonic statistical properties, $\mathcal{U}_{(q)}(h(1))$ plays the role of the spectrum generating algebra ¹⁰ of the q -deformed oscillator. ¹⁻³ Following Refs. [12, 13], we do not impose the $g^2 = 1$ restriction. The universal \mathcal{R} matrix of the $\mathcal{U}_{(q)}(h(1))$ algebra has also been obtained. ¹⁴

Two Hopf algebras \mathcal{U} and \mathcal{A} are in duality ¹⁸ if there exists a doubly-nondegenerate bilinear form

$$\langle , \rangle : (\mathbf{a}, \mathbf{u}) \rightarrow \langle \mathbf{a}, \mathbf{u} \rangle \quad \forall \mathbf{a} \in \mathcal{A}, \quad \forall \mathbf{u} \in \mathcal{U}, \quad (2.3)$$

such that, for $(\mathbf{a}, \mathbf{b}) \in \mathcal{A}$, $(\mathbf{u}, \mathbf{v}) \in \mathcal{U}$,

$$\begin{aligned} \langle \mathbf{a}, \mathbf{u}\mathbf{v} \rangle &= \langle \Delta_{\mathcal{A}}(\mathbf{a}), \mathbf{u} \otimes \mathbf{v} \rangle, \quad \langle \mathbf{a}\mathbf{b}, \mathbf{u} \rangle = \langle \mathbf{a} \otimes \mathbf{b}, \Delta_{\mathcal{U}}(\mathbf{u}) \rangle, \\ \langle \mathbf{a}, \mathbb{I}_{\mathcal{U}} \rangle &= \epsilon_{\mathcal{A}}(\mathbf{a}), \quad \langle \mathbb{I}_{\mathcal{A}}, \mathbf{u} \rangle = \epsilon_{\mathcal{U}}(\mathbf{u}), \quad \langle \mathbf{a}, S_{\mathcal{U}}(\mathbf{u}) \rangle = \langle S_{\mathcal{A}}(\mathbf{a}), \mathbf{u} \rangle. \end{aligned} \quad (2.4)$$

Let the ordered monomials $E_{k\ell m} = a^{\dagger k} \tilde{N}^\ell a^m$, $(k, \ell, m) \in (0, 1, 2, \dots)$ be the basis elements of the $\mathcal{U}_{(q)}(h(1))$ algebra obeying the multiplication and the induced coproduct rules given by

$$E_{k\ell m} E_{k'\ell' m'} = \sum_{pqr} f_{k\ell m \ k'\ell' m'}^{pqr} E_{pqr}, \quad \Delta(E_{k\ell m}) = \sum_{\substack{pqr \\ p'q'r'}} g_{k\ell m}^{pqr \ p'q'r'} E_{pqr} \otimes E_{p'q'r'}. \quad (2.5)$$

The basis elements $e^{k\ell m}$ of the dual Hopf algebra $\mathcal{F}un_{(q)}(H(1))$ follows the relation

$$\langle e^{k\ell m}, E_{k'\ell' m'} \rangle = \delta_{k'}^k \delta_{\ell'}^\ell \delta_{m'}^m. \quad (2.6)$$

In particular, the generating elements of the $\mathcal{F}un_{(q)}(H(1))$ algebra, defined as $x = e^{100}$, $y = e^{001}$ and $z = e^{010}$, satisfy the following duality structure:

$$\langle x, a^\dagger \rangle = 1, \quad \langle z, \tilde{N} \rangle = 1, \quad \langle y, a \rangle = 1. \quad (2.7)$$

The duality condition (2.4) requires the basis set $e^{k\ell m}$ to obey the multiplication and coproduct rules given below:

$$e^{pqr} e^{p'q'r'} = \sum_{k\ell m} g_{k\ell m}^{pqr p'q'r'} e^{k\ell m}, \quad \Delta(e^{pqr}) = \sum_{\substack{k\ell m \\ k'\ell' m'}} f_{k\ell m}^{pqr} f_{k'\ell' m'}^{p'q'r'} e^{k\ell m} \otimes e^{k'\ell' m'}. \quad (2.8)$$

To derive the Hopf properties of the dual $\mathcal{F}un_{\langle q \rangle}(H(1))$ algebra, we, therefore, need to extract the structure constants defined in (2.5). Towards this end we note that the induced coproduct map of the elements $E_{k\ell m}$ may be obtained via (2.2):

$$\begin{aligned} \Delta(E_{k\ell m}) &= \Delta(a^\dagger)^k \Delta(\tilde{N})^\ell \Delta(a)^m \\ &= (a^\dagger \otimes q^{\alpha\tilde{N}/2} + \exp(-i\pi\tilde{N}) q^{-\alpha\tilde{N}/2} \otimes a^\dagger)^k (\tilde{N} \otimes I + I \otimes \tilde{N})^\ell \times \\ &\quad \times (a \otimes q^{\alpha\tilde{N}/2} + \exp(i\pi\tilde{N}) q^{-\alpha\tilde{N}/2} \otimes a)^m, \end{aligned} \quad (2.9)$$

where we have used $(-1)^{\pm\tilde{N}} = \exp(\pm i\pi\tilde{N})$. Employing (2.9) we now obtain a set of structure constants:

$$\begin{aligned} g_{k\ell m}^{100 \ 001} &= \delta_{k1} \delta_{\ell0} \delta_{m1}, \quad g_{k\ell m}^{001 \ 100} = \delta_{k1} \delta_{\ell0} \delta_{m1}, \\ g_{k\ell m}^{010 \ 100} &= \delta_{k1} \delta_{\ell1} \delta_{m0} - \left(\frac{\alpha}{2} \ln q + i\pi\right) \delta_{k1} \delta_{\ell0} \delta_{m0}, \\ g_{k\ell m}^{100 \ 010} &= \delta_{k1} \delta_{\ell1} \delta_{m0} + \frac{\alpha}{2} \ln q \delta_{k1} \delta_{\ell0} \delta_{m0}, \\ g_{k\ell m}^{010 \ 001} &= \delta_{k0} \delta_{\ell1} \delta_{m1} - \left(\frac{\alpha}{2} \ln q - i\pi\right) \delta_{k0} \delta_{\ell0} \delta_{m1}, \\ g_{k\ell m}^{001 \ 010} &= \delta_{k0} \delta_{\ell1} \delta_{m1} + \frac{\alpha}{2} \ln q \delta_{k0} \delta_{\ell0} \delta_{m1}. \end{aligned} \quad (2.10)$$

The above structure constants immediately yield the algebraic relations obeyed by the generators of the $\mathcal{F}un_{\langle q \rangle}(H(1))$ algebra:

$$[x, y] = 0, \quad [z, x] = -(\alpha \ln q + i\pi) x, \quad [z, y] = -(\alpha \ln q - i\pi) y. \quad (2.11)$$

A representation of the above Lie algebra with complex structure constants may be easily obtained in terms of harmonic oscillators $\{\mathfrak{a}_i, \mathfrak{a}_i^\dagger \mid i = (1, 2)\}$ obeying the algebra $[\mathfrak{a}_i, \mathfrak{a}_j^\dagger] = \delta_{ij}$, $[\mathfrak{a}_i, \mathfrak{a}_j] = 0$, $[\mathfrak{a}_i^\dagger, \mathfrak{a}_j^\dagger] = 0$, $(i, j) = (1, 2)$:

$$x = \mathfrak{a}_1, \quad y = \mathfrak{a}_2, \quad z = \alpha \ln q (\mathfrak{a}_1^\dagger \mathfrak{a}_1 + \mathfrak{a}_2^\dagger \mathfrak{a}_2) + i\pi (\mathfrak{a}_1^\dagger \mathfrak{a}_1 - \mathfrak{a}_2^\dagger \mathfrak{a}_2). \quad (2.12)$$

Proceeding towards constructing the coproduct maps of the generating elements of the dual $\mathcal{F}un_{\langle q \rangle}(H(1))$ algebra we notice that the defining properties (2.8) provide the necessary recipe:

$$\begin{aligned} \Delta(x) &= \sum_{\substack{k\ell m \\ k'\ell' m'}} f_{k\ell m}^{100} f_{k'\ell' m'}^{p'q'r'} e^{k\ell m} \otimes e^{k'\ell' m'}, \\ \Delta(z) &= \sum_{\substack{k\ell m \\ k'\ell' m'}} f_{k\ell m}^{010} f_{k'\ell' m'}^{p'q'r'} e^{k\ell m} \otimes e^{k'\ell' m'}, \\ \Delta(y) &= \sum_{\substack{k\ell m \\ k'\ell' m'}} f_{k\ell m}^{001} f_{k'\ell' m'}^{p'q'r'} e^{k\ell m} \otimes e^{k'\ell' m'}. \end{aligned} \quad (2.13)$$

The relevant structure constants obtained via (2.5) are listed below:

$$\begin{aligned}
f_{k\ell m k' \ell' m'}^{100} &= \delta_{k1} \delta_{\ell0} \delta_{m0} \delta_{k'0} \delta_{\ell'0} \delta_{m'0} + \sigma_{m+1} \delta_{k0} \delta_{k'm+1} \delta_{\ell'0} \delta_{m'0}, \\
f_{k\ell m k' \ell' m'}^{010} &= \delta_{k0} \delta_{m0} \delta_{k'0} \delta_{m'0} (\delta_{\ell1} \delta_{\ell'0} + \delta_{\ell0} \delta_{\ell'1}) + \frac{2\alpha \ln q}{q - q^{-1}} \sigma_m \delta_{k0} \delta_{\ell0} \delta_{k'm} \delta_{\ell'0} \delta_{m'0}, \\
f_{k\ell m k' \ell' m'}^{001} &= \delta_{k0} \delta_{\ell0} \delta_{m0} \delta_{k'0} \delta_{\ell'0} \delta_{m'1} + \sigma_{k'+1} \delta_{k0} \delta_{\ell0} \delta_{m'k'+1} \delta_{m'0}, \\
\sigma_1 &= 1, \quad \sigma_m (>1) = \prod_{k=1}^{m-1} \sum_{\ell=0}^{k-1} (-1)^\ell [(k-\ell)\alpha]_q. \tag{2.14}
\end{aligned}$$

The coproduct maps of the dual generators may now be explicitly obtained *à la* (2.13) provided the basis elements $e^{k\ell m}$ of the dual $\mathcal{F}un_{(q)}(H(1))$ algebra are known. We complete this task subsequently.

As the dual algebra $\mathcal{F}un_{(q)}(H(1))$ is finitely generated, we may start with the generators (x, y, z) and obtain all dual basis elements $e^{k\ell m}$, $(k, \ell, m) \in (0, 1, 2, \dots)$ by successively applying the multiplication rule given in the first equation in (2.8). The necessary structure constants may be read from the relation (2.5) of the $\mathcal{U}_{(q)}(h(1))$ algebra. In the procedure described below we maintain the operator ordering of the monomials as $x^k z^\ell y^m$, $(k, \ell, m) \in (0, 1, 2, \dots)$. The product rule

$$e^{100} e^{p00} = \sum_{k\ell m} g_{k\ell m}^{100 p00} e^{k\ell m} \tag{2.15}$$

and the explicit evaluation of the structure constant

$$g_{k\ell m}^{100 p00} = \{k\}_{q^\alpha} \delta_{k p+1} \delta_{\ell0} \delta_{m0}, \quad \{n\}_q = \frac{q^{n/2} - (-1)^n q^{-n/2}}{q^{1/2} + q^{-1/2}} \tag{2.16}$$

obtained from the second equation in (2.5) immediately provide

$$e^{k00} = \frac{x^k}{\{k\}_{q^\alpha}!}, \quad \{n\}_q! = \prod_{\ell=1}^n \{\ell\}_q, \quad \{0\}_q! = 1. \tag{2.17}$$

Employing another product rule

$$e^{p00} e^{010} = \sum_{k\ell m} g_{k\ell m}^{p00 010} e^{k\ell m} \tag{2.18}$$

and the value of the relevant structure constant

$$g_{k\ell m}^{p00 010} = (r+1) \delta_{kp} \delta_{\ell r+1} \delta_{m0} + \frac{\alpha p}{2} \ln q \delta_{kp} \delta_{\ell r} \delta_{m0} \tag{2.19}$$

obtained in the aforesaid way we produce the following result:

$$e^{k\ell 0} = \frac{x^k}{\{k\}_{q^\alpha}!} \frac{\left(z - \frac{\alpha}{2} k \ln q\right)^\ell}{\ell!}. \tag{2.20}$$

Continuing the above process of building of the dual basis set we use the product rule

$$e^{prs} e^{001} = \sum_{k\ell m} g_{k\ell m}^{prs001} e^{k\ell m} \quad (2.21)$$

and the value of the corresponding structure constant

$$g_{k\ell m}^{prs001} = \{m\}_{q^\alpha} \sum_{j=0}^r \frac{1}{j!} \left(-\frac{\alpha}{2} \ln q + i\pi \right)^j \delta_{kp} \delta_{\ell r-j} \delta_{m s+1} \quad (2.22)$$

obtained via (2.5). This finally leads us to the complete construction of the dual basis element:

$$e^{k\ell m} = \frac{x^k}{\{k\}_{q^\alpha}!} \frac{\left(z - \frac{\alpha}{2} (k-m) \ln q - im\pi \right)^\ell}{\ell!} \frac{y^m}{\{m\}_{q^\alpha}!}. \quad (2.23)$$

Combining our results in (2.13), (2.14) and (2.23), we now provide the promised coproduct structure of the generators of the $\mathcal{F}un_{\langle q \rangle}(H(1))$ algebra:

$$\begin{aligned} \Delta(x) &= x \otimes \mathbb{I} + \sum_{m=0}^{\infty} (-1)^m q^{m\alpha/2} \sigma_{m+1} \exp(z) \frac{y^m}{\{m\}_{q^\alpha}!} \otimes \frac{x^{m+1}}{\{m+1\}_{q^\alpha}!}, \\ \Delta(z) &= z \otimes \mathbb{I} + \mathbb{I} \otimes z + \frac{2\alpha \ln q}{q - q^{-1}} \sum_{m=1}^{\infty} \sigma_m \frac{y^m}{\{m\}_{q^\alpha}!} \otimes \frac{x^m}{\{m\}_{q^\alpha}!}, \\ \Delta(y) &= \mathbb{I} \otimes y + \sum_{m=0}^{\infty} q^{-m\alpha/2} \sigma_{m+1} \frac{y^{m+1}}{\{m+1\}_{q^\alpha}!} \otimes \frac{x^m}{\{m\}_{q^\alpha}!} \exp(z). \end{aligned} \quad (2.24)$$

Algebraic simplifications allow us to express the coproduct maps of the above generators more succinctly:

$$\begin{aligned} \Delta(x) &= x \otimes \mathbb{I} + \sum_{m=0}^{\infty} (-1)^m \left(\frac{q^\alpha + 1}{q - q^{-1}} \right)^m \exp(z) y^m \otimes x^{m+1}, \\ \Delta(z) &= z \otimes \mathbb{I} + \mathbb{I} \otimes z + \frac{2\alpha \ln q}{q - q^{-1}} \sum_{m=1}^{\infty} \frac{1}{\{m\}_{q^\alpha}} \left(\frac{q^{\alpha/2} + q^{-\alpha/2}}{q - q^{-1}} \right)^{m-1} y^m \otimes x^m, \\ \Delta(y) &= \mathbb{I} \otimes y + \sum_{m=0}^{\infty} \left(\frac{1 + q^{-\alpha}}{q - q^{-1}} \right)^m y^{m+1} \otimes x^m \exp(z). \end{aligned} \quad (2.25)$$

With the aid of the result (2.25) we may explicitly demonstrate that the coproduct map is a homomorphism of the algebra (2.11): namely,

$$\begin{aligned} [\Delta(x), \Delta(y)] &= 0, & [\Delta(z), \Delta(x)] &= -(\alpha \ln q + i\pi) \Delta(x), \\ [\Delta(z), \Delta(y)] &= -(\alpha \ln q - i\pi) \Delta(y). \end{aligned} \quad (2.26)$$

The coassociativity constraint

$$(\text{id} \otimes \Delta) \circ \Delta(\mathcal{X}) = (\Delta \otimes \text{id}) \circ \Delta(\mathcal{X}) \quad \forall \mathcal{X} \in (x, y, z) \quad (2.27)$$

may also be established by using the following identity:

$$\begin{aligned} \exp(\Delta(z)) &= (\exp(z) \otimes \mathbb{I}) \prod_{m=1}^{\infty} \mathcal{P}_m (\mathbb{I} \otimes \exp(z)) \\ \mathcal{P}_m &= \exp \left((-1)^m \frac{[m\alpha]_q}{m\{m\}_{q^\alpha}} \left(\frac{q^{\alpha/2} + q^{-\alpha/2}}{q - q^{-1}} \right)^{m-1} y^m \otimes x^m \right). \end{aligned} \quad (2.28)$$

The counit map of the generators of the $\mathcal{F}un_{\langle q \rangle}(H(1))$ algebra reads as

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 0. \quad (2.29)$$

The antipode map of the dual generators follows from the last equation in (2.4). We quote the results here:

$$\begin{aligned} S(x) &= \sum_{m=0}^{\infty} (-1)^m q^{(m+2)\alpha/2} \sigma_{m+1} \frac{x^{m+1}}{\{m+1\}_{q^\alpha}!} \exp(-(m+1)z) \frac{y^m}{\{m\}_{q^\alpha}!}, \\ S(z) &= -z + \frac{2\alpha \ln q}{q - q^{-1}} \sum_{m=1}^{\infty} \sigma_m \frac{x^m}{\{m\}_{q^\alpha}!} \exp(-mz) \frac{y^m}{\{m\}_{q^\alpha}!}, \\ S(y) &= \sum_{m=0}^{\infty} q^{-(m+2)\alpha/2} \sigma_{m+1} \frac{x^m}{\{m\}_{q^\alpha}!} \exp(-(m+1)z) \frac{y^{m+1}}{\{m+1\}_{q^\alpha}!}. \end{aligned} \quad (2.30)$$

In an order by order calculation we may verify that the above antipode map is an anti-homomorphism of the algebra (2.11), and the necessary Hopf constraint holds:

$$m \circ (S \otimes \text{id}) \circ \Delta(\mathcal{X}) = m \circ (\text{id} \otimes S) \circ \Delta(\mathcal{X}) = \epsilon(\mathcal{X}) \mathbb{I} \quad \forall \mathcal{X} \in (x, y, z), \quad (2.31)$$

where m is the multiplication map. This completes our construction of the Hopf algebra $\mathcal{F}un_{\langle q \rangle}(H(1))$ dually related to the generalized boson algebra $\mathcal{U}_{\langle q \rangle}(h(1))$.

Our explicit listing of the complete set of dual basis elements in (2.23) allows us to obtain *à la* Fronsdal and Galindo¹⁸ the universal \mathcal{T} matrix:

$$\mathcal{T} = \sum_{k,\ell,m} e^{k\ell m} \otimes E_{k\ell m} \equiv \mathcal{T}_{e,E}. \quad (2.32)$$

The notion of the universal \mathcal{T} matrix is a key feature capping the Hopf duality structure. Consequently, the duality relations (2.4) may be concisely expressed¹⁸ in terms of the \mathcal{T} matrix as

$$\begin{aligned} \mathcal{T}_{e,E} \mathcal{T}_{e',E} &= \mathcal{T}_{\Delta(e),E}, & \mathcal{T}_{e,E} \mathcal{T}_{e,E'} &= \mathcal{T}_{e,\Delta(E)}, \\ \mathcal{T}_{\epsilon(e),E} &= \mathbb{I}, & \mathcal{T}_{e,\epsilon(E)} &= \mathbb{I}, & \mathcal{T}_{S(e),E} &= \mathcal{T}_{e,S(E)}, \end{aligned} \quad (2.33)$$

where e and e' (E and E') refer to the two identical copies of $\mathcal{F}un_{\langle q \rangle}(H(1))$ ($\mathcal{U}_{\langle q \rangle}(h(1))$) algebra.

As both the Hopf algebras in our case are finitely generated the universal \mathcal{T} matrix may now be obtained as an operator valued function in a closed form:

$$\mathcal{T} = {}_{\times}^{\times} \mathcal{E} \text{xp}_{q^\alpha} \left(x \otimes a^\dagger q^{-\alpha \tilde{N}/2} \right) \exp(z \otimes \tilde{N}) \mathcal{E} \text{xp}_{q^\alpha} \left(y \otimes (-1)^{-\tilde{N}} q^{\alpha \tilde{N}/2} a \right) {}_{\times}^{\times}, \quad (2.34)$$

where $\mathcal{E}\text{xp}_q(\mathcal{X}) = \sum_{m=0}^{\infty} \frac{\mathcal{X}^n}{\{n\}_q!}$. The operator ordering has been explicitly indicated above. The universal \mathcal{T} matrix, as evidenced in (2.34), may be viewed¹⁸ as the appropriate quantum group generalization of the familiar exponential map relating a Lie algebra with the corresponding Lie group. We note that the deformed exponential in (2.34) is different from that in Ref. [18]. The universal \mathcal{T} matrix given in (2.34) is endowed with a group-like coproduct rule and it is characterized by *noncommuting* parameters (x, y, z) in a representation-independent way.

III. Coherent states of a single-node system and their completeness

As a prelude to our subsequent construction of the entangled coherent states in a bipartite composite system, we, in the present section, study the completeness of the coherent states in a single-node system possessing the deformed Heisenberg symmetry $\mathcal{U}_{\langle q \rangle}(h(1))$ defined in (2.1) and (2.2). A Fock-type representation of the algebra $\mathcal{U}_{\langle q \rangle}(h(1))$ is given by^{12, 13}

$$\begin{aligned} |n\rangle &= \frac{1}{\sqrt{(n)_{\alpha,\beta}!}} (a^\dagger)^n |0\rangle \quad n \in (0, 1, 2, \dots), \quad a|0\rangle = 0, \\ a|n\rangle &= \sqrt{(n)_{\alpha,\beta}} |n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{(n+1)_{\alpha,\beta}} |n+1\rangle, \quad (-1)^N |n\rangle = (-1)^n |n\rangle, \\ (n)_{\alpha,\beta} &= (q^{\alpha/2} + q^{-\alpha/2})^{-1} ([n\alpha + \beta - \alpha/2]_q + (-1)^{n+1} [\beta - \alpha/2]_q), \\ (n)_{\alpha,\beta}! &= \prod_{\ell=1}^n (\ell)_{\alpha,\beta}, \quad (0)_{\alpha,\beta}! = 1, \quad \langle n|n' \rangle = \delta_{nn'}. \end{aligned} \quad (3.1)$$

For a single-node system the coherent state is defined as

$$a|\zeta\rangle = \zeta|\zeta\rangle, \quad \zeta \in \mathbb{C}. \quad (3.2)$$

The normalized coherent state reads

$$|\zeta\rangle = \frac{1}{\sqrt{\exp_{\alpha,\beta}(|\zeta|^2)}} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{(n)_{\alpha,\beta}!}} |n\rangle = \frac{1}{\sqrt{\exp_{\alpha,\beta}(|\zeta|^2)}} \exp_{\alpha,\beta}(\zeta a^\dagger) |0\rangle, \quad (3.3)$$

where the deformed exponential is given by

$$\exp_{\alpha,\beta}(\mathcal{X}) = \sum_{n=0}^{\infty} \frac{\mathcal{X}^n}{(n)_{\alpha,\beta}!}. \quad (3.4)$$

The coherent states (3.3) have nonvanishing inner products, and, therefore, are not orthogonal:

$$\langle \zeta' | \zeta \rangle = \frac{\exp_{\alpha,\beta}(\overline{\zeta'} \zeta)}{\sqrt{\exp_{\alpha,\beta}(|\zeta'|^2) \exp_{\alpha,\beta}(|\zeta|^2)}}. \quad (3.5)$$

Assuming the completeness of the discrete basis states

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I}, \quad (3.6)$$

we now prove that the coherent states $|\zeta\rangle$ possess a resolution of identity with a positive definite integration measure in the complex plane. For constructing this measure we proceed by defining a generalized Gamma function suited to our purpose:

$$\Gamma_{\alpha,\beta}(z) = \prod_{\ell=1}^{\infty} \frac{(\ell)_{\alpha,\beta}}{(z + \ell - 1)_{\alpha,\beta}}, \quad \Gamma_{\alpha,\beta}(z + 1) = (z)_{\alpha,\beta} \Gamma_{\alpha,\beta}(z), \quad \Gamma_{\alpha,\beta}(n + 1) = (n)_{\alpha,\beta}!. \quad (3.7)$$

In the limit $\alpha \rightarrow 2, \beta \rightarrow 1, q \rightarrow 1$, the deformed Gamma function $\Gamma_{\alpha,\beta}(z)$ reduces to its classical partner $\Gamma(z)$. Analytic continuation of $(n)_{\alpha,\beta}$ defined in (3.1) for noninteger arguments may be done in two possible ways ($n \rightarrow z \Rightarrow (-1)^n \rightarrow \exp(\pm i\pi z)$) yielding results related to each other by complex conjugation:

$$(z)_{\alpha,\beta}^{(\pm)} = (q^{\alpha/2} + q^{-\alpha/2})^{-1} ([z\alpha + \beta - \alpha/2]_q - \exp(\pm i\pi z) [\beta - \alpha/2]_q). \quad (3.8)$$

The generalized $\Gamma_{\alpha,\beta}(z)$ functions (3.7) corresponding to the said two analytic continuations are referred to as $\Gamma_{\alpha,\beta}^{(\pm)}(z)$. Omitting the superscripts here, we note that the singularity structure of the generalized Gamma function may be derived from the following iterated relation:

$$\Gamma_{\alpha,\beta}(z) = \frac{\Gamma_{\alpha,\beta}(z + n + 1)}{(z)_{\alpha,\beta} (z + 1)_{\alpha,\beta} \cdots (z + n)_{\alpha,\beta}}. \quad (3.9)$$

The two analytic continuations given in (3.8), in the limit $\varepsilon \rightarrow 0$, yield

$$(\varepsilon)_{\alpha,\beta}^{(\pm)} = \varepsilon \left(\alpha [[\beta - \alpha/2]] \frac{\ln q}{q - q^{-1}} \mp i \varpi [\beta - \alpha/2]_q \right), \quad (3.10)$$

where $[[\mathcal{X}]] = \frac{q^{\mathcal{X}} + q^{-\mathcal{X}}}{q^{\alpha/2} + q^{-\alpha/2}}$, $\varpi = \frac{\pi}{q^{\alpha/2} + q^{-\alpha/2}}$. Using (3.9) and (3.10) the singularity structure of the conjugate functions $\Gamma_{\alpha,\beta}^{(\pm)}(z)$ in the neighbourhood $z = -n + \varepsilon$ may be obtained as

$$\Gamma_{\alpha,\beta}^{(\pm)}(-n + \varepsilon) = \frac{1}{(\varepsilon)_{\alpha,\beta}^{(\pm)} (n)_{-\alpha,\beta-\alpha}!}, \quad (3.11)$$

where we have used $(-n)_{\alpha,\beta} = (n)_{-\alpha,\beta-\alpha}$. As we are interested in the positive definiteness of the integration measure the said analytic continuation must be done in a *symmetric way by taking an average of the two complex conjugate functions*:

$$\Gamma_{\alpha,\beta}^{sym}(z) = (\Gamma_{\alpha,\beta}^{(+)}(z) + \Gamma_{\alpha,\beta}^{(-)}(z))/2. \quad (3.12)$$

The singularity of the above symmetrized deformed Gamma function is obtained by using (3.11):

$$\begin{aligned} \Gamma_{\alpha,\beta}^{sym}(-n + \varepsilon) &= (\Gamma_{\alpha,\beta}^{(+)}(-n + \varepsilon) + \Gamma_{\alpha,\beta}^{(-)}(-n + \varepsilon))/2 \\ &= \varepsilon^{-1} \frac{\mathcal{P}}{(n)_{-\alpha,\beta-\alpha}!}, \end{aligned} \quad (3.13)$$

where

$$\mathcal{P} = \frac{\alpha [[\beta - \alpha/2]] \ln q/(q - q^{-1})}{(\alpha [[\beta - \alpha/2]] \ln q/(q - q^{-1}))^2 + \varpi^2 [\beta - \alpha/2]_q^2}. \quad (3.14)$$

Parallel to the undeformed Gamma function, our $\Gamma_{\alpha,\beta}^{sym}(z)$ also possess, as evident from above, simple poles at $z = 0, -1, -2, \dots$. Keeping in mind the above singularity structure of the generalized $\Gamma_{\alpha,\beta}^{sym}(z)$, we now obtain a resolution of the identity via coherent states $|\zeta\rangle$ in the form

$$\int d\mu(\zeta) |\zeta\rangle\langle\zeta| = \mathbb{I}, \quad (3.15)$$

where the integration measure $d\mu(\zeta)$ is determined below. Using the polar decomposition $\zeta = \rho \exp(i\theta)$ with our construction of the coherent state (3.3), we integrate the angular variable θ to obtain

$$\exp_{\alpha,\beta}(\rho^2) \int_0^{2\pi} \frac{d\theta}{2\pi} |\zeta\rangle\langle\zeta| = \sum_{n=0}^{\infty} \frac{\rho^{2n}}{(n)_{\alpha,\beta}!} |n\rangle\langle n|. \quad (3.16)$$

Multiplying both sides of the above equation by a yet to be determined function $F(\rho)$, and integrating over the entire complex ζ plane, we get

$$\int d^2\zeta \exp_{\alpha,\beta}(|\zeta|^2) F(|\zeta|) |\zeta\rangle\langle\zeta| = \sum_{n=0}^{\infty} \frac{\mathcal{I}_n}{(n)_{\alpha,\beta}!} |n\rangle\langle n|, \quad (3.17)$$

where $d^2\zeta = (2\pi)^{-1} \rho d\rho d\theta$, and \mathcal{I}_n represents the Mellin transform of the function $F(\rho)$:

$$\mathcal{I}_n = \int_0^{\infty} d\rho \rho^{2n+1} F(\rho). \quad (3.18)$$

If we now choose the transform \mathcal{I}_n in (3.17) as

$$\mathcal{I}_n = (n)_{\alpha,\beta}!, \quad (3.19)$$

it immediately follows that by the virtue of completeness relation (3.6) of the discrete basis states $|n\rangle$, the rhs in (3.17) reduces to identity operator:

$$\int d^2\zeta \exp_{\alpha,\beta}(|\zeta|^2) F(|\zeta|) |\zeta\rangle\langle\zeta| = \mathbb{I}. \quad (3.20)$$

The function $F(\rho)$ defined by the Stieltjes moment relation may now be explicitly obtained in terms of an inverse Mellin transform as

$$F(\rho) = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} dz \rho^{-2z} (z-1)_{\alpha,\beta}! = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} dz \rho^{-2z} \Gamma_{\alpha,\beta}^{sym}(z). \quad (3.21)$$

In the second equation we have used, as explained earlier in the context of (3.12), a symmetrized analytic continuation of the deformed factorial. Using the singularity structure (3.13) we now explicitly evaluate the previously undetermined measure function $F(\rho)$

via the contour integral (3.21) as the integral vanishes exponentially as $|z| \rightarrow \infty$ on the left-half plane:

$$F(|\zeta|) = \frac{2\alpha [[\beta - \alpha/2]] \ln q/(q - q^{-1})}{(\alpha [[\beta - \alpha/2]] \ln q/(q - q^{-1}))^2 + \varpi^2 [\beta - \alpha/2]_q^2} \exp_{-\alpha, \beta - \alpha}(|\zeta|^2). \quad (3.22)$$

To conclude about the positivity of the measure, we, as it is evident from (3.17) and (3.22), need to study the positivity of the deformed exponential $\exp_{\alpha, \beta}(\mathcal{X})$ for arbitrary real arguments. We demonstrate this by adopting a method previously used in another context by Quesne.²⁷ The generalized exponential function $\exp_{\alpha, \beta}(\mathcal{X})$ may be expressed as a product of ordinary exponentials:

$$\exp_{\alpha, \beta}(\mathcal{X}) = \exp\left(\sum_{k=1}^{\infty} c_k \mathcal{X}^k\right), \quad (3.23)$$

where the coefficients c_k obey a linear recurrence relation

$$c_k = \frac{1}{(k)_{\alpha, \beta}!} - \frac{1}{k} \sum_{\ell=1}^{k-1} \frac{\ell}{(k-\ell)_{\alpha, \beta}!} c_{\ell}, \quad c_1 = \frac{1}{(1)_{\alpha, \beta}}. \quad (3.24)$$

The above triangular set of linear equations may be solved up to any arbitrary order, and the first few coefficients are written below:

$$\begin{aligned} c_1 &= \frac{1}{[\beta]_q}, & c_2 &= \frac{1}{[\alpha]_q [\beta]_q [[\beta + \alpha/2]]} - \frac{1}{2[\beta]_q^2}, \\ c_3 &= \frac{1}{[\alpha]_q [\beta]_q [\alpha + \beta]_q [[3\alpha/2]] [[\beta + \alpha/2]]} - \frac{1}{[\alpha]_q [\beta]_q^2 [[\beta + \alpha/2]]} + \frac{1}{3[\beta]_q^3}. \end{aligned} \quad (3.25)$$

A consequence of the product structure (3.23) is that the deformed exponential $\exp_{\alpha, \beta}(\mathcal{X})$ is a positive definite quantity for real arguments. The above discussion leads us to infer that the measure function obtained via (3.15), (3.17), and (3.22)

$$d\mu(\zeta) = d^2\zeta \frac{2\alpha [[\beta - \alpha/2]] \ln q/(q - q^{-1})}{(\alpha [[\beta - \alpha/2]] \ln q/(q - q^{-1}))^2 + \varpi^2 [\beta - \alpha/2]_q^2} \exp_{\alpha, \beta}(|\zeta|^2) \exp_{-\alpha, \beta - \alpha}(|\zeta|^2) \quad (3.26)$$

is a positive definite quantity for $\alpha > 0$.

IV. Bipartite composite systems and entangled coherent states

The Hopf coalgebraic structure of the $\mathcal{U}_{(q)}(h(1))$ algebra given in (2.2) is expected to play a qualitatively important role in describing the symmetry properties of many body systems. Keeping this picture in mind, we, in the present section, introduce the normalized coherent states of the $\mathcal{U}_{(q)}(h(1))$ algebra in the case of a bipartite composite system. The bipartite coherent states may be defined as

$$\Delta(a) |\widehat{\zeta}\rangle = \zeta |\widehat{\zeta}\rangle, \quad \zeta \in \mathbb{C}, \quad (4.1)$$

where the noncocommutative coproduct structure $\Delta(a)$ is given in (2.2). Expanding of the state (4.1) in the tensored basis of the number states

$$|\widehat{\zeta}\rangle = \sum_{n,m} c_{n,m} |n\rangle \otimes |m\rangle. \quad (4.2)$$

we obtain a double-indexed recurrence relations for the coefficients $c_{n,m}$:

$$c_{n+1,m} \sqrt{(n+1)_{\alpha,\beta}} q^{(m\alpha+\beta)/2} + c_{n,m+1} (-1)^{n+\beta/\alpha} \sqrt{(m+1)_{\alpha,\beta}} q^{-(n\alpha+\beta)/2} = \zeta c_{n,m}. \quad (4.3)$$

While describing the solution of the recurrence relation (4.3), we, for the purpose of comparison, stay as close as possible to the construction (3.3) of the coherent states of single-node systems. Pursuing this approach we consider the ansatz

$$c_{n,m} = \frac{\zeta_1^n \zeta_2^m}{\sqrt{(n)_{\alpha,\beta}! (m)_{\alpha,\beta}!}} g_{n,m}, \quad (\zeta_1, \zeta_2) \in \mathbb{C} \quad (4.4)$$

and redefine the parameters as follows

$$\frac{\zeta_1}{\zeta} q^{\beta/2} = \rho_1, \quad (-1)^{\beta/\alpha} \frac{\zeta_2}{\zeta} q^{-\beta/2} = \rho_2 \quad (4.5)$$

to obtain a simpler recurrence relation satisfied by the coefficients $g_{n,m}$:

$$\rho_1 q^{m\alpha/2} g_{n+1,m} + (-1)^n \rho_2 q^{-n\alpha/2} g_{n,m+1} = g_{n,m}. \quad (4.6)$$

We proceed towards solving the above recurrence relation by considering the coefficients $g_{n,m}$ as elements of a matrix. A little reflection then shows that given the elements of the first row we can obtain all other elements by employing (4.6) successively. Assuming the boundary condition

$$g_{0,m} = d_m, \quad 0 < d_m \leq 1 \quad (4.7)$$

the solution of the recurrence relation (4.6) may be found by inspection. We quote the result:

$$\begin{aligned} g_{n,m} &= \frac{q^{-nm\alpha/2}}{\rho_1^n} \sum_{k=0}^n (-1)^{k(k+1)/2} \binom{n}{k}_{q^{-\alpha}} \rho_2^k q^{-k(k-1)\alpha/2} d_{m+k}, \\ \binom{n}{k}_q &= \frac{\binom{n}{q}_q!}{\binom{k}{q}_q! \binom{n-k}{q}_q!}, \quad \binom{n}{q}_q! = \prod_{\ell=1}^n \binom{\ell}{q}_q, \quad \binom{n}{q}_q = \frac{1 - (-1)^n q^n}{1 + q}. \end{aligned} \quad (4.8)$$

For special choices of the boundary coefficients d_m , the matrix elements $g_{n,m}$ may be expressed in closed form. For instance, choosing $d_m = \delta^m$, $0 < \delta \leq 1$, we obtain

$$g_{n,m} = q^{-nm\alpha/2} \frac{\delta^m}{\rho_1^n} (\delta \rho_2; -q^{-\alpha})_n, \quad (a; q)_n = \prod_{\ell=1}^n (1 - a q^{\ell-1}). \quad (4.9)$$

At this point it is useful to examine (4.9) in the $q \rightarrow 1$ limit. Retaining the previous notations, the coefficients $g_{n,m}$ in the $q \rightarrow 1$ limit are found to assume the form

$$g_{2\ell,m} = \delta^m \left(\frac{1 - \delta^2 \rho_2^2}{\rho_1^2} \right)^\ell, \quad g_{2\ell+1,m} = \delta^m \frac{1 - \delta \rho_2}{\rho_1} \left(\frac{1 - \delta^2 \rho_2^2}{\rho_1^2} \right)^\ell. \quad (4.10)$$

It is apparent from (4.10) that the coherent state of the composite bipartite system, in the $q \rightarrow 1$ limit, may be factorized in the states of the single-node subsystems.

Returning to the deformed case ($q \neq 1$), we combine (4.2), (4.4), (4.5) and (4.9) to finally obtain the bipartite coherent state as

$$|\widehat{\zeta}\rangle = \sum_{n,m} q^{-n m \alpha/2} \frac{\delta^m \zeta_1^n \zeta_2^m}{\rho_1^n \sqrt{(n)_{\alpha,\beta}! (m)_{\alpha,\beta}!}} (\delta \rho_2; -q^{-\alpha})_n |n\rangle \otimes |m\rangle. \quad (4.11)$$

In the above construction of the bipartite coherent state $|\widehat{\zeta}\rangle$, the complex variables ρ_1 and ρ_2 (or, equivalently, ζ_1 and ζ_2) enter as arbitrary parameters. The norm of the bipartite coherent state (4.11) may now be readily obtained as

$$\mathcal{N} \equiv \langle \widehat{\zeta} | \widehat{\zeta} \rangle = \sum_{n,m} |c_{n,m}|^2 = \sum_n \frac{q^{-n\beta} |\zeta|^{2n}}{(n)_{\alpha,\beta}!} |(\delta \rho_2; -q^{-\alpha})_n|^2 \exp_{\alpha,\beta} \left(\delta^2 |\zeta_2|^2 q^{-n\alpha} \right). \quad (4.12)$$

Using the definition of the deformed exponential function (3.4) the norm has been expressed as a single sum. Its convergence in various domains may be tested in a straightforward way. For instance, in the region $q > 1, (\alpha, \beta) > 0$, the sum is convergent, and, therefore, the norm (4.12) is finite. The normalized coherent state $\mathcal{N}^{-1/2} |\widehat{\zeta}\rangle$ of the bipartite system obeying the $\mathcal{U}_{(q)}(h(1))$ Hopf symmetry may be readily obtained from (4.11) and (4.12). The most remarkable property of the bipartite coherent state (4.11) is its naturally entangled structure for a nonclassical ($q \neq 1$) value of the deformation parameter. *The summand in (4.11) include the factor $q^{-n m \alpha/2}$, which forbids factorization of the coherent state of the composite system into quantum states of single-node components.*

V. Conclusion

In conclusion we briefly mention the possibilities of further development of the topics discussed here. In connection with our derivation of the closed form expression of the universal \mathcal{T} matrix capping the Hopf duality structure of the $\mathcal{U}_{(q)}(h(1))$ and the $\mathcal{F}un_{(q)}(H(1))$ algebras, we mention the followings. Transfer matrices of integrable models are finite dimensional representations of the operator valued universal \mathcal{T} matrix. As Calogero-Sutherland type of models are known¹³ to have close kinship with the $\mathcal{U}_{(q)}(h(1))$ algebra, our universal \mathcal{T} matrix may be of use in finding new deformations of these models. Moreover, $\mathcal{U}_{(q)}(h(1))$ algebra is¹⁰ the bosonized version of the Hopf superalgebra $\mathcal{U}_q(osp(1|2))$. Our universal \mathcal{T} matrix may provide direct clues on the derivation of as yet unknown universal \mathcal{T} matrix of the $\mathcal{U}_q(osp(1|2))$ algebra. This is likely to be useful in constructing $\mathcal{U}_q(osp(1|2))$ based integrable models.

Entanglement of states is the key feature in quantum information processing, such as quantum teleportation,²⁸ quantum key distribution²⁹ and so on. Our work raises the

interesting possibility that the composite systems such as anyons³⁰ for instance, may be naturally equipped for implementing entangled states. If the composite system allows for a variation of the deformation parameter q , a ‘switching mechanism’ for entanglement may be developed. Lastly, the multipartite systems subject to $\mathcal{U}_{\langle q \rangle}(h(1))$ symmetry show new levels of entanglement. We will return to this topic in a future work.

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